Quality-Reliability Chain Modeling for System-Reliability Analysis of Complex Manufacturing Processes

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***Abstract*—System reliability of a manufacturing process should address effects of both the manufacturing system (MS) component reliability, and the product quality. In a multi-station manufacturing process (MMP), the degradation of MS components at an upstream station can cause the deterioration of the downstream product quality. At the same time, the system component reliability can be affected by the deterioration of the incoming product quality of upstream stations. This kind of quality & reliability interaction characteristics can be observed in many manufacturing processes such as machining, assembly, and stamping. However, there is no available model to describe this complex relationship between product quality, and MS component reliability. This paper, considering the unique complex characteristics of MMP, proposes a new concept of quality & reliability chain (QR-Chain) effect to describe the complex propagation relationship of the interaction between MS component reliability, and product quality across all stations. Based on this, a general QR-chain model for MMP is proposed to integrate the product quality with the MS component reliability information for system reliability analysis. For evaluation of system reliability, both the exact analytic solution, and a simpler upper bound solution are provided. The upper bound is proved to be equal to the exact solution if the product quality does not have self-improvement, which is generally true in many MMP. Therefore, the developed QR-chain model, and its upper bound solution can be applied to many MMP.**

***Index Terms*—Doubly stochastic process, multi-station manufacturing process, quality & reliability dependency, quality & reliability chain, system reliability evaluation.**

ACRONYMS[1]

DOE	design of experiments
i.i.d.	s-independent & identically distributed
MMP	multi-station manufacturing process
MS	manufacturing system
NHPP	nonhomogeneous Poisson process
Pdf	probability distribution function
PQC	product quality characteristic
QR-chain	quality & reliability chain
r.v.	random variable
TE	tooling element

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[1]The singular and plural of an acronym are always spelled the same.

NOTATION

$\boldsymbol{a}$	design specification vector for product quality
$\mathbf{A} \otimes \mathbf{B}$	Kronecker product of $\mathbf{A}$ & $\mathbf{B}$
$\mathbf{A}_k$, $\mathbf{G}_k$	possibly time-varying, known matrices of appropriate dimension used in the degradation model
$[\mathbf{b}]_i$	element $\#i$ of vector $\mathbf{b}$
$F_{\mathbf{X}}$	cdf of a random variable $\mathbf{X}$
g	number of s-independent random increments in the component degradation model
l	number of noise-variables in the system
m	number of product quality characteristics
n	number of process-variables in an MS
p	number of MS components
$q_j(t)$	quality index $\#j$
t	number of operation cycles, which is an operating time index
t_K	production mission time (reliability evaluation lifetime)
$\mathbf{x} \circ \mathbf{y}$	Hadamard product of two vectors $\mathbf{x}$ & $\mathbf{y}$ ($\mathbf{z} = \mathbf{x} \circ \mathbf{y} \Rightarrow [\mathbf{z}]_i = [\mathbf{x}]_i \cdot [\mathbf{y}]_i$)
$\mathbf{X}(t) \in R^n$	vector of process-variables at time t, with $\mathbf{X}(t_0) \sim N(\boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0)$
$\mathbf{Y}(t)$	$[Y_1(t)\, Y_2(t)\, \ldots\, Y_m(t)]^T$, vector of PQC at time t
η_j	constants in the process model
$\boldsymbol{\alpha}_j, \boldsymbol{\beta}_j$	vectors characterizing the effects of $\mathbf{X}(t)$, and $\mathbf{z}_t$ in the process model
$\boldsymbol{\Gamma}_j$	a matrix characterizing the effects of the interactions between $\mathbf{X}(t)$, and $\mathbf{z}_t$
$\boldsymbol{\varepsilon}_k \in R^g$	noise term in the MS component degradation model; $\{\boldsymbol{\varepsilon}_k, k \geq 1\}$ are i.i.d., and s-normal, with $\boldsymbol{\varepsilon}_k \sim N(\boldsymbol{\mu}_\varepsilon, \mathbf{Q})$
$\lambda_i(t)$	catastrophic failure rate of component i at operation t
λ_{0i}	initial fixed failure rate
$\mathrm{unif}(a, b)$	the distribution of equal probability of 0.5 at a & b
$\boldsymbol{\mu}(k)$	mean vector of $\mathbf{X}(t_k)$
$\boldsymbol{\mu}_0$	initial mean of process-variable at t_0
$\boldsymbol{\Sigma}(k)$	covariance matrix of $\mathbf{X}(t_k)$
$\boldsymbol{\Sigma}_0$	initial covariance of process-variable at t_0

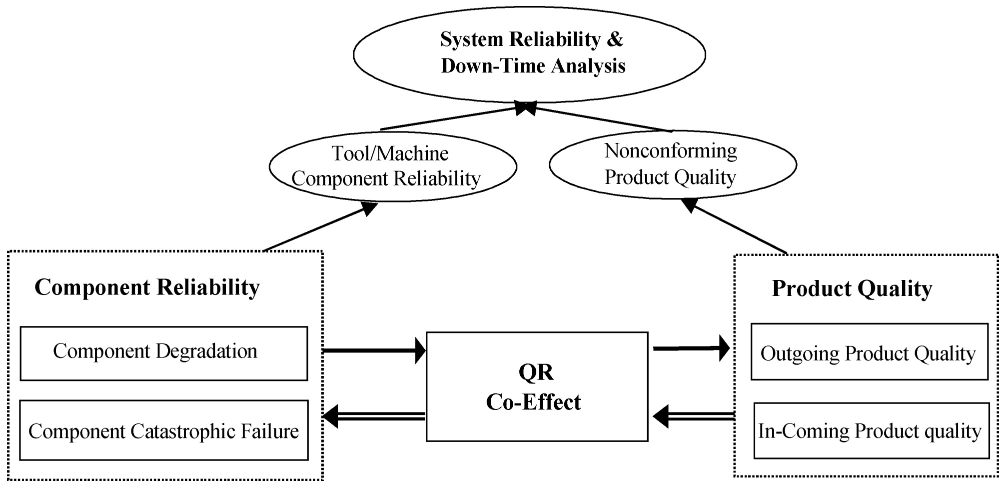

Fig. 1. QR-co-effect between product quality, and MS component reliability.

I. Introduction

A. Problem Statement

An MMP is a process to produce a product through a series of stations, which are used in various industry sectors. Examples of MMP include multi-assembly stations in automotive body assembly, transfer or progressive dies in stamping processes, and multiple pattern lithography operations in semiconductor manufacturing. In these processes, unanticipated machine/tool failures or degradations can cause unanticipated process downtime or nonconforming products. Therefore, the development of a system reliability model for a manufacturing process needs to consider both the effects of product quality, and MS component reliability.

System reliability is generally defined as the probability that a system performs its intended function under operating conditions for a specified period of time [1]. The intended function of a manufacturing process should consider not only the MS up-time, but also the produced product quality. Many research efforts were conducted to study various reliability models & evaluation methods [2]–[5], where system failures are determined only by MS component failures due to their malfunctions, or their degradations beyond the maximum acceptable amount of tool wear. The dependency of the MS reliability on the product quality has been overlooked on papers about system reliability modeling of a manufacturing process.

For a single-station manufacturing process, the concept of the interaction between product quality & MS component reliability, the QR-co-effect, was proposed in [6] for manufacturing process reliability modeling. Fig. 1 shows that the MS component degradation affects the outgoing product quality. For example, in a tapping process, the wear of a tap (taken as component degradation) deteriorates the quality of the tapped threads. Meanwhile the incoming raw material/part quality can affect the MS component reliability, such as degradation rate, and catastrophic failure rate. For example, the straightness of the hole on the raw part affects the breakage rate of the tap in a tapping process. Based on this model, [6] proves that there is an overestimation of the manufacturing process reliability if the QR-co-effect is not considered in the system reliability modeling.

The QR-co-effect model for a single-station developed in [6] assumes the event of nonconforming product quality is s-independent of component catastrophic failures. However, this assumption does not hold in an MMP due to the dependent propagation of product quality across stations. Therefore, the development of an effective model to describe the quality & reliability dependency, and its propagation throughout all stations, is one of the major concerns for system reliability evaluation in an MMP.

B. QR-Chain Effect in an MMP

Definitions:

- Manufacturing system—a system consisting of several tools & equipment to perform the designed operations in an MMP, and produce products as its end item.
- Component/MS component—a physical part of an MS, such as a tool, equipment, or a machine. In this paper, component means the component of an MS, rather than the component of a product. Examples of components include locators in assembly processes, cutting tools & drills in machining processes, and dies in stamping processes.
- Component catastrophic failure—failure of components immediately leading to downtime of the component. Examples of component catastrophic failures include breakage of tools such as locators & cutting tools.
- Product—both the incoming & outgoing work-pieces at each station of an MMP, which includes the intermediate parts, and the final product.
- Product quality—the variation of the dimensional quality characteristics, such as size, straightness, and orientation of a drilled hole in machining processes, size of a burr, and dimension of a formed part in stamping processes. In this paper, we focus on dimensional integrity of manufactured products.
- Component reliability information—includes component catastrophic failure information such as catastrophic failure rate, and component degradation information such as wear rate.

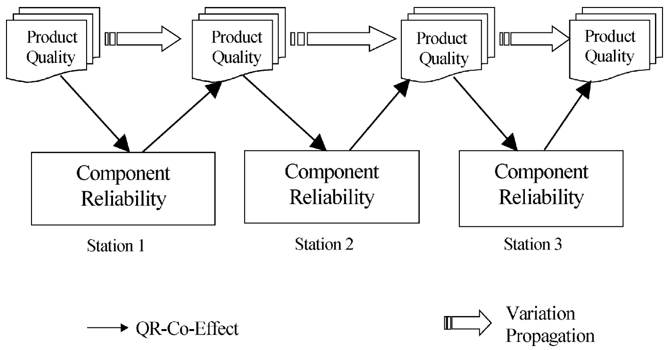

Fig. 2. General concepts of the QR-chain in MMP.

- System catastrophic failure—system failure due to component catastrophic failures.
- System failure due to nonconforming products—an event that the produced products are out of specifications. For MMP, the specifications can be generally assigned to the intermediate or final products.
- System reliability of an MMP—the probability that neither the system catastrophic failure nor the failure due to nonconforming products occurs during a specific period of time.

In an MMP, each station consists of multiple components. To simplify the problem, all components in an MMP are assumed to be in series; the catastrophic failure of any component leads to the system catastrophic failure. All the developed models & methodologies in this paper can be easily adapted to more general hybrid series/parallel systems.

The product variation is propagated in an MMP [7], [8]. The final product quality is affected by the accumulation or stack up of all variations generated at previous stations. Considering the QR-co-effect at each station, the variation propagation in product quality leads to the propagation of the interaction between the MS component reliability, and the product quality (Fig. 2), which is the QR-chain effect. The QR-chain effect is a unique characteristic of an MMP to be studied in this paper.

The QR-chain effect can be observed in many MMP. These examples are given here to illustrate the common characteristics of the QR-chain effect in various manufacturing processes:

1. Machining processes: The QR-Chain effect can be observed in a cylinder head machining process. For an example from [9], as shown in Fig. 3, there are two stations consisting of drilling a hole in a cylinder head (Station I), and then tapping a thread on it afterwards (Station II). In Station I, the material properties of the incoming work-piece (taken as the product quality) have an important impact on the wear & breakage rate of the drill (taken as the MS component reliability). The drill condition further impacts the quality of the hole drilled in this station in terms of size, straightness, orientation, etc. In the next tapping station (Station II), those drilled hole quality characteristics of Station I are essential factors affecting the thread quality of Station II, and the breakage rate of the tap. Therefore, the QR-co-effect at Station I has propagated to Station II. Because the stochastic degradation of the drill affects not only the breakage rate of the tap but also the conformity of the final product quality, the catastrophic failures of the tap, and the system failure due to nonconforming products are s-dependent.

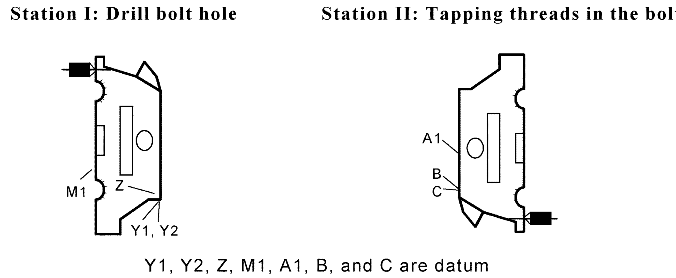

Fig. 3. QR-chain in a machining process.

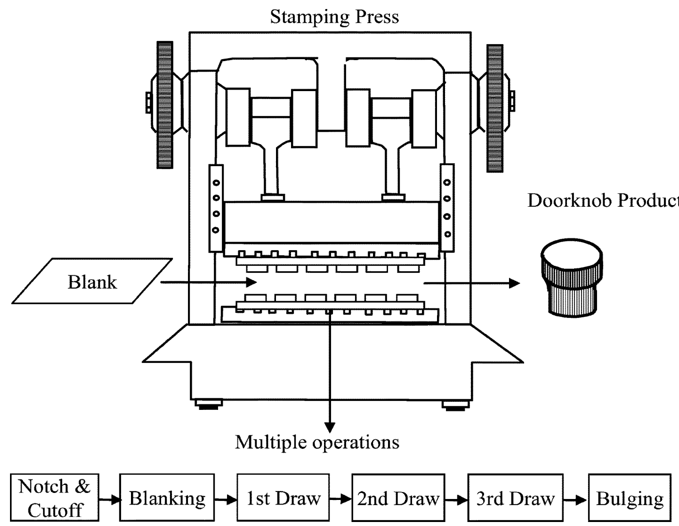

Fig. 4. QR-Chain in a stamping process with a transfer die.

2. Transfer or progressive die stamping processes: Similar (to Example 1) QR-chain effects can be observed in a stamping process with transfer or progressive dies, where multiple stations are used to form a part. Fig. 4 shows a doorknob forming process consisting of 6 stations in a sequence:
   i. notch & cutoff,
   ii. blanking,
   iii. the 1st draw,
   iv. the 2nd draw,
   v. the 3rd draw, and
   vi. bulging [10].

   In general, the product quality in previous stations impacts the current die or tool degradation, which further impacts the quality of products produced in the current station. As an example, the blanking die worn-out at Station 2 (taken as component reliability) generates burr on the edge of the part (taken as the outgoing product quality of Station 2). In the following draw stations from Stations 3 to 5, the size of the burr (taken as incoming product quality) has an important impact on the draw die wear & breakage rate (taken as component reliability), and a further impact on the dimension & surface quality of the formed part. In the last bulging station, the combined effects of the formed part dimension & the size of burr on the part surface have impacts on the rubber tool wear rate (rubber functions as a die) at the bulging station. Thus, a clear QR-Chain effect can be observed in this process. Because the stochastic degradation of the blanking die at station 2 affects not only the breakage rate of the draw dies in the following stations but also the conformity of the quality of the final formed work piece, the failures of the draw dies & the system failure due to nonconforming products are s-dependent.

From these MMP examples, the common characteristics of the QR-chain effect are summarized as:

a. The quality of the outgoing products at a station depends on the reliability of system components at the current station, as well as the quality of incoming products produced by the previous stations. On the other hand, the reliability of the MS components at the current station can be affected by incoming product quality from the previous stations.
b. The deterioration of the outgoing product quality is caused by the degradation of the MS components. The dependent propagation of product quality across stations causes the s-dependency of the catastrophic failures of MS components, and the system failure due to nonconforming products. This is a special characteristic for an MMP compared with a single station process.

C. Literature Review

The QR-co-effect has been paid more attention in recent engineering literature. The effect of tool faults on product quality has been quantitatively modeled for various manufacturing processes [7], [11]. On the other hand, the impact of incoming product quality on the failure of the MS component has been observed in [12]. These engineering models can be used to obtain the process model of Section II-A of this paper. However, no system reliability model of a manufacturing process has been built to capture the QR-chain effect.

The popular system reliability models in the literature include the fault tree ([13], [14]), Markov chain ([15]–[17]), and Petri net ([18]). However, these models have the following disadvantages in studying the QR-chain effect:

1. they cannot model the *continuous* changes of component wear states; and
2. they cannot capture the *continuous* relationship between the wear of multiple tools & the deviation/variation of multiple quality characteristics.

Because of these two reasons, none of the existing models can be directly used to evaluate system reliability for an MMP with the QR-chain effect.

Over the past few years, some literature devoted to the evolution of a relatively new class of failure models is based on physics-of-failure, and the characteristics of operating environment. A comprehensive review of this kind of models is in [19]. Among these models, [20] & [21] study the item degradation with diffusion processes. The item is considered as failed when its degradation state reaches a threshold level. Another type of model studies the covariate induced failure rate processes [22], [23]. In these models, the failure rate of an item is influenced by a covariate whose evolution is also a stochastic process. These two types of models can be used to model the component degradation, and the component catastrophic failure of a manufacturing process, by treating the product quality deterioration as a covariate process influencing the component catastrophic failures. However, none of these physics-of-failure-based models can be used to model an MMP with the QR-chain effect because:

1. Most of these physics-of-failure-based models are developed for the single-component failure rather than the failure of a multi-component, multi-station system.
2. None of these physics-of-failure-based models captures the product quality information. However, as mentioned in Section I-B the failure of an MMP should be defined based on product quality rather than the degradation state of a single component.

Thus, a model describing the relationship between component performance & product quality, and a scheme to assess the product quality are the most essential elements in modeling of MMP with the QR-chain effect.

This paper introduces a QR-chain model for MMP to integrate the MS component degradation & catastrophic failure information with the product quality information. Based on this model, a general methodology is presented for system reliability analysis in MMP. Section II proposes a new system reliability model: the QR-chain model, for general MMP. Section III performs the system reliability evaluation to obtain an analytic solution for the proposed QR-chain model, and an upper bound of the system reliability is derived. Section IV uses an example to illustrate the analysis procedures, and the effectiveness of the proposed methodology.

II. QR-Chain Modeling

Based on the characteristics of the QR-chain effect summarized in Section I, this section discusses the modeling procedures describing the relationship between component performance & product quality, the degradation process of components, the assessment criteria of product quality, and the impact of the product quality on the component catastrophic failures.

Assumptions:

1. Discrete-part manufacturing processes are considered, where the number of operation cycles is treated as the time index.
2. The component degradation can be modeled by a discrete approximation to a diffusion process with mean of the wear linearly dependent on the component degradation state, and its variance constant.
3. The wear within a time interval has a Gaussian distribution.
4. Without loss of generality, the component is in the ideal state if $\mathbf{X}(t) = \mathbf{0}$.
5. All components have nondecreasing wear.
6. The product quality can be assessed by the s-expected squared deviation of the PQC from the target.

7. Pr{a system component fails during the next operation time | it still works at the current operation time} is assumed to be proportional to the linear combination of the squared deviations of certain PQC.
8. The effects of different PQC on component catastrophic failure are s-independent.

A. Relationship Between Component Performance & Product Quality

The product quality in an MMP is generally affected by the state of multiple MS components. The important impact of the MS component performance on the product quality has been demonstrated in [24], where the component state is described by process-variables. Another class of variables affecting product quality is the random noise that is not determined by the component state, called the noise-variables. Examples of noise-variables include random variations of raw material quality, and random environmental variations. In general, noise-variables randomly change from one operation time to the next. Considering further the interaction between the process-variables & the noise-variables, the following general linear model is assumed for the PQC $Y_j(t)$, $j = 1, 2, \ldots, m$, which is called as the *process model* in this paper

$$Y_j(t) = \eta_j + \boldsymbol{\alpha}_j^T \times \mathbf{X}(t) + \boldsymbol{\beta}_j^T \times \mathbf{z}_t + \mathbf{X}(t)^T \times \boldsymbol{\Gamma}_j \times \mathbf{z}_t, \quad j = 1, 2, \ldots, m \quad (1)$$

where $\mathbf{z}_t \equiv [z_{1t}, z_{2t}, \ldots, z_{lt}]^T \in R^l$ is the vector of noise-variables, with mean $\mathrm{E}(\mathbf{z}_t)$ & covariance matrix $\mathrm{Cov}(\mathbf{z}_t)$ independent of the time index.

Remarks: The proposed process model (1) is called the response model in robust parameter design [25], [26]. In parameter design, the process-variables in (1) are often called the main effect of the control factors, and the noise-variables are called the main effect of the noise factors. Based on the effect ordering principle in parameter design, the main effect of the control factors, the main effect of the noise factors, and the control by noise interactions are the most important effects on product quality. Thus the linear combination of these three effects, as in (1), is widely used in robust parameter design to model the PQC. When the physical process knowledge is available, the process model (1) can be obtained based on the specific physical process models. Otherwise, it can be generally obtained by using DOE because the process model (1) has the same structure as the response model in robust parameter design. The term 'control factor' is not used in this paper because the process-variables are exactly determined by their inherent component degradation states, rather than by externally controlled process parameter setups.

B. System Component Degradation

In general, $\mathbf{X}(t)$ in (1) change over time due to system component degradation. Let the time axis be divided into contiguous & uniform intervals of length h, and the successive endpoints of the intervals be denoted by $h, 2h, 3h, \ldots, kh, \ldots$. The process-variable vector $\mathbf{X}(t_k) \in R^n$ represents the degradation state of system components at time $t_k \equiv kh$, $k = 1, 2, \ldots$. The production mission time is $t_K = K \cdot h$. A well-known single item degradation model [20] is

$$X(t_{k+1}) - X(t_k) = \mu\left(X(t_k)\right) + \sigma\left(X(t_k)\right) \cdot \varepsilon_k \quad (2)$$

where $\{\varepsilon_k, k \geq 1\}$ is a sequence of i.i.d. r.v.. When $\{\varepsilon_k, k \geq 1\}$ is s-normally distributed, (2) becomes a discrete approximation to a diffusion process.

In [4], the attention is restricted to a simpler class of the diffusion processes where $\mu(\cdot)$ & $\sigma(\cdot)$ are independent of X & k. For a multivariate version of (2), the following model can be obtained based on Assumption #2:

$$\mathbf{X}(t_{k+1}) = \mathbf{A}_k \times \mathbf{X}(t_k) + \mathbf{G}_k \times \boldsymbol{\varepsilon}_k, \quad k = 0, 1, 2, \ldots \quad (3)$$

Equation (3) generally describes a Gauss-Markov process. The assumption of the s-normal distribution of $\boldsymbol{\varepsilon}_k$ here is due to its mathematical tractability, and the enormous literature on diffusion processes & its applications in degradation modeling. For a large scale MMP with many operation cycles during each time interval, the s-normal distribution assumption can also be justified by the central limit theorem if the wear increments of each operation cycle during a time interval are s-independent. From Assumptions 4 & 5, for any $1 \leq j \leq n$, the $\Pr\{[\mathbf{X}(t_0)]_j < 0\}$, and $\Pr\{[\mathbf{X}(t_{k+1}) - \mathbf{X}(t_k)]_j < 0\}$ can be ignored. In fact, decreasing (or negative) wear is not meaningful in many applications [19].

C. Product Quality Assessment & System Failure Due to Nonconforming Products

Based on the derivation in Appendix A1, the quality index $q_j(t)$ under the given component degradation state, $\mathbf{X}(t_k)$, can be written in the following form:

$$q_j\left(t|\mathbf{X}(t_k)\right) = \mathbf{X}(t_k)^T \times \mathbf{B}_j \times \mathbf{X}(t_k) + d_j, \quad j = 1, 2, \ldots, m, t_k \leq t < t_{k+1}. \quad (4)$$

$\mathbf{B}_j$ in (4) is positive semi-definite, and $d_j \geq 0$.

For each quality index, there is a threshold value based on the product design specifications. Let the threshold for the quality-index $\#j$ be a_j, and E_t^q be defined as the event that all quality indexes are within the specification by time t, i.e.

$$E_t^q \equiv \bigcap_{j=1}^{m} \left(q_j(\tau) \leq a_j, \forall 0 \leq \tau \leq t\right). \quad (5)$$

In (5), $q_j(\tau)(t_i \leq \tau < t_{i+1})$ can be obtained by taking the expectation on $q_j(t|\mathbf{X}(t_i))$, i.e.,

$$q_j(\tau) = \underset{\mathbf{X}(t_i)}{\mathrm{E}}\left[q_j\left(t|\mathbf{X}(t_i)\right)\right].$$

In practice, a_j in (5) can be determined by the product quality specification & the following process capability ratio, C_{pm}, which is widely used in quality engineering [27]

$$C_{pm} = \frac{\mathrm{USL} - \mathrm{LSL}}{6\sqrt{\mathrm{MSE}}}. \quad (6)$$

USL-LSL is the tolerance range for a product quality characteristic, and MSE is the expected squared deviation. For example, in automotive assembly processes, the specification limit for dimensional deviations of the body-in-white (an automotive body without doors, hood, fenders, and trunk lid) is 2 mm. To achieve a process capability with $C_{pm} = 1$, from (6) it can be easily seen that MSE cannot be greater than $(2/6)^2\ \mathrm{mm}^2$. Therefore, a_j should be selected as $(2/6)^2\ \mathrm{mm}^2$ for this example.

D. Component Catastrophic Failure, and Its Induced System Catastrophic Failure

From Assumption 7

$$\begin{aligned}&\Pr\{\text{component } i \text{ fails at operation } t+1| \\ &\quad \text{it works at operation } t, \mathbf{Y}(t)\} \\ &\qquad = \lambda_{0i} + \mathbf{s}_i^T \times ((\mathbf{Y}(t)-\boldsymbol{\gamma}) \circ (\mathbf{Y}(t)-\boldsymbol{\gamma})), \\ &\qquad i = 1,2,\ldots,p \\ &\quad \mathbf{s}_i \in R^m. \end{aligned} \tag{7}$$

$\mathbf{s}_i$ is called the QR-coefficient in this paper, and has only non-negative elements; $\boldsymbol{\gamma} \equiv [\gamma_1\ \gamma_2\ \ldots\ \gamma_m]^T$. With the failure rate being minimum at $\mathbf{Y}(t) = \boldsymbol{\gamma}$ (quality characteristics right on the target), and with Assumption 8, the quadratic relationship in (7) can be considered as a second order approximation of a general functional relationship based on the Taylor series. By taking the s-expectation on the noise-variables in $\mathbf{Y}(t)$, and using the definition of the failure rate, $\lambda_i(t)$ can be written as

$$\begin{aligned}\lambda_i(t) &= \Pr\{\text{component } i \text{ fails during } (t,t+1)| \\ &\qquad \text{it works at } t, \mathbf{X}(t_k)\} \\ &= \mathrm{E}\left[\lambda_{0\mathrm{i}} + \mathbf{s}_i^T \times ((\mathbf{Y}(t)-\boldsymbol{\gamma})\circ(\mathbf{Y}(t)-\boldsymbol{\gamma})\,|\mathbf{X}(t_k)\right] \\ &= \lambda_{0i} + \mathbf{s}_i^T \times \mathrm{E}[(\mathbf{Y}(t)-\boldsymbol{\gamma})\circ(\mathbf{Y}(t)-\boldsymbol{\gamma})\,|\mathbf{X}(t_k)] \\ &= \lambda_{0i} + \mathbf{s}_i^T \times \mathbf{q}\,(t|\mathbf{X}(t_k)) \\ \mathbf{q}(t|\mathbf{X}(t_k)) &= [q_1(t|\mathbf{X}(t_k))\ q_2\,(t|\mathbf{X}(t_k))\ \ldots\ q_m(t|\mathbf{X}(t_k))]^T. \end{aligned} \tag{8}$$

Define E_t^c as the event that catastrophic failures never occurred at any of the p system components by time t. Then there is no system catastrophic failure by time t if E_t^c holds.

III. System Reliability Evaluation

Based on the definition of system reliability of an MMP with the QR-chain effect, the system reliability at the production mission time, t_K is $R(t_K) = \Pr\{E_{t_K}^q \cap E_{t_K}^c\}$.

A. Challenges in System Reliability Evaluation

The complex interactions among various elements of the QR-chain model lead to the following challenges in system reliability evaluation:

- *s-dependency between E_t^q & E_t^c*: Both E_t^q & E_t^c depend on the system component degradation in an MMP. So they are not s-independent to each other in general: $R(t) = \Pr\{E_t^q \cap E_t^c\} \neq \Pr\{E_t^q\} \cdot \Pr\{E_t^c\}$.
- *s-dependency among system component degradation*: Generally the degradation of a system component depends not only on its current state, but also on that of other system components. So the degradation processes of system components are not s-independent.
- *s-dependency among catastrophic failures of system components*: The catastrophic failures of system components at each station of an MMP can depend on the incoming product quality. The same quality characteristic of the incoming product can affect multiple system components. Different quality characteristics are generally not s-independent of each other, because they all depend on the system component degradation processes in previous stations. As a result, the catastrophic failures of system components are not s-independent of each other.
- *Doubly stochastic property*: From (4) & (8), the $\lambda_i(t)$ depends on another stochastic process $\{\mathbf{X}(t_k), k = 1,2,\ldots\}$. In the literature, the failure process with its failure rate depending on another stochastic process is called "doubly stochastic Poisson process" [28]. Thus the failure process of each manufacturing system component in the QR-chain model is a doubly stochastic Poisson process.

B. System Reliability Evaluation of MMP

Three steps are proposed to evaluate the QR-chain model:

Step 1. Conditioning on the degradation path of each component, the component catastrophic failure process, which is a doubly stochastic Poisson process, becomes a NHPP. And the dependency of E_t^q & E_t^c can also be removed.

Step 2. Un-condition on the degradation paths by implementing s-expectation to the conditional system reliability calculated in Step 1.

Step 3. Reorganize the integrand in (11) obtained in Step 2 into the form of the pdf of a Gaussian r.v. in order to efficiently apply Monte Carlo simulation methods.

1) Step 1: A conditional system reliability is evaluated in this step by conditioning on the degradation path $\{\mathbf{X}(t_k), k = 1,2,\ldots\}$, or $\mathbf{X}_K \equiv [\mathbf{X}^T(t_0)\ \mathbf{X}^T(t_1)\ \ldots\ \mathbf{X}^T(t_K)]^T$. Conditioning on $\mathbf{X}_K$, $E_{t_K}^q$ & $E_{t_K}^c$ are s-independent. So the conditional system reliability

$$R(t_K|\mathbf{X}_K) = \Pr\left\{E_{t_K}^c|\mathbf{X}_K\right\} \cdot \Pr\left\{E_{t_K}^q|\mathbf{X}_K\right\}. \tag{9}$$

$\Pr\{E_{t_K}^c|\mathbf{X}_K\}$ & $\Pr\{E_{t_K}^q|\mathbf{X}_K\}$ can be calculated by Results 1 & 2 shown next.

Result 1: Let $\mathbf{d} \equiv [d_1\ d_2\ \ldots\ d_m]^T$, and $c \equiv \sum_{i=1}^p (\lambda_{0i} + \mathbf{s}_i^T \times \mathbf{d})$. There exists a positive semi-definite matrix $\mathbf{U}_K$, such that

$$\Pr\left\{E_{t_K}^c|\mathbf{X}_K = \mathbf{x}_K\right\} = \exp\left(-\left(c \cdot t_K + \mathbf{x}_K^T \times \mathbf{U}_K \times \mathbf{x}_K\right)\right).$$

◀The proof of Result 1 is in Appendix A2.

Result 2: $\boldsymbol{\Omega}$ is a domain in $R^{(K+1)\times n}$ s.t. $\mathbf{x}_K \in \boldsymbol{\Omega} \Leftrightarrow \bigcap_{k=0}^K \bigcap_{j=1}^m \{\mathbf{x}^T(t_k) \times \mathbf{B}_j \times \mathbf{x}(t_k) \leq a_j - d_j\}$. Let

$$I_K \equiv \begin{cases} 1, & \text{if } \mathbf{x}_K \in \boldsymbol{\Omega} \\ 0, & \text{otherwise} \end{cases} \qquad \Pr\left\{E_{t_K}^q|\mathbf{X}_K = \mathbf{x}_{1K}\right\} = I_K.$$

◀This result is obvious from the definition of the system failure due to nonconforming products, and (4).

From Result 1, Result 2, and (9);

$$R(t_K|\mathbf{X}_K = \mathbf{x}_K) = \exp\left(-\left(c \cdot t_K + \mathbf{x}_K^T \times \mathbf{U}_K \times \mathbf{x}_K\right)\right) \cdot I_{\mathrm{K}}. \tag{10}$$

2) Step 2: Un-conditioning on $\mathbf{X}_K$ by taking the s-expectation of (10), we have

$$\begin{aligned} R(t_K) = &\int_{\mathbf{x}_K \in \mathbf{\Omega}} \exp\left(-\left(c \cdot t_K + \mathbf{x}_K^T \times \mathbf{U}_K \times \mathbf{x}_K\right)\right) \\ &\times \frac{1}{(2\pi)^{\frac{n\cdot(K+1)}{2}} \cdot |\mathbf{\Sigma}_K|^{\frac{1}{2}}} \\ &\times \exp\left(-\frac{1}{2}(\mathbf{x}_K - \boldsymbol{\mu}_K)^T \times \mathbf{\Sigma}_K^{-1} \times (\mathbf{x}_K - \boldsymbol{\mu}_K)\right) d\mathbf{x}_K \\ = &\exp(-c \cdot t_K) \int_{\mathbf{x}_K \in \mathbf{\Omega}} \frac{1}{(2\pi)^{\frac{n\cdot(K+1)}{2}} \cdot |\mathbf{\Sigma}_K|^{\frac{1}{2}}} \\ &\times \exp\left\{-\left(\mathbf{x}_K^T \times \mathbf{U}_K \times \mathbf{x}_K + \frac{1}{2}(\mathbf{x}_K - \boldsymbol{\mu}_K)^T \right.\right. \\ &\left.\left. \times \mathbf{\Sigma}_K^{-1} \times (\mathbf{x}_K - \boldsymbol{\mu}_K)\right)\right\} d\mathbf{x}_K. \end{aligned} \tag{11}$$

Please refer to Appendix A3 for the detailed derivation of (11), and the definition of the notations used in (11).

3) Step 3: Transform the integrand in (11) to the form of the pdf of another multivariate Gaussian r.v., $\widetilde{\mathbf{X}}_K$, based on Lemma 1.

Lemma 1: There exists a positive definite matrix $\widetilde{\mathbf{\Sigma}}_K$, a vector $\widetilde{\boldsymbol{\mu}}_K$, and a scalar $s_K > 0$ such that

$$\begin{aligned} &\mathbf{x}_K^T \times \mathbf{U}_K \times \mathbf{x}_K + \frac{1}{2}(\mathbf{x}_K - \boldsymbol{\mu}_K)^T \times \mathbf{\Sigma}_K^{-1} \times (\mathbf{x}_K - \boldsymbol{\mu}_K) \\ &\quad = \frac{1}{2}(\mathbf{x}_K - \widetilde{\boldsymbol{\mu}}_K)^T \times \widetilde{\mathbf{\Sigma}}_K^{-1} \times (\mathbf{x}_K - \widetilde{\boldsymbol{\mu}}_K) + s_K. \end{aligned} \tag{12}$$

Lemma 1 is proved in Appendix A4. From Lemma 1, the integrand in (11) can be transformed into the form of a multivariate Gaussian pdf as in Result 3.

Result 3: Let $\widetilde{\mathbf{X}}_K$ be an $n(K+1)$ dimension Gaussian r.v. whose pdf is

$$\begin{aligned} f_{\widetilde{\mathbf{X}}_K}(\mathbf{x}_K) = &\frac{1}{(2\pi)^{\frac{n(K+1)}{2}} \cdot |\widetilde{\mathbf{\Sigma}}_K|^{\frac{1}{2}}} \\ &\times \exp\left\{-\frac{1}{2} \cdot (\mathbf{x}_K - \widetilde{\boldsymbol{\mu}}_K)^T \times \widetilde{\mathbf{\Sigma}}_K^{-1} \times (\mathbf{x}_K - \widetilde{\boldsymbol{\mu}}_K)\right\}. \end{aligned}$$

Then the system reliability can be written as

$$R(t_K) = \exp(-c \cdot t_K) \cdot \frac{\exp(-s_K)}{|\mathbf{\Sigma}_K|^{\frac{1}{2}}} \cdot |\widetilde{\mathbf{\Sigma}}_K|^{\frac{1}{2}} \cdot \int_{\mathbf{x}_K \in \mathbf{\Omega}} dF_{\widetilde{\mathbf{X}}_K}(\mathbf{x}_K). \tag{13}$$

Equation (13) can be proved by reorganizing the exponential part of the integrand in (11) based on Lemma 1.

The multidimensional integral in (13) can be calculated by the following Monte Carlo simulation method:

Generate N_s Gaussian r.v. with mean $\widetilde{\boldsymbol{\mu}}_K$, and variance $\widetilde{\mathbf{\Sigma}}_K$, among which N_0 generated r.v. fall in the quality constraint $\mathbf{\Omega}$. Then N_0/N_s is an estimate of the integral $\int_{\mathbf{x}_K \in \mathbf{\Omega}} dF_{\widetilde{\mathbf{X}}_K}(\mathbf{x}_K)$. Alternative approaches include other multidimensional numerical integration methods, such as multidimensional Gaussian quadratures [29]. To be effective, however, these methods usually require a simple region of integration, and low integral dimension. Due to the complexity of the region of integration $\mathbf{\Omega}$, and the typically high dimension of the integral, we suggest to use the Monte Carlo simulation method. The efficiency of the proposed Monte Carlo simulation method will be discussed as follows.

Let $\pi \equiv \int_{\mathbf{x}_K \in \mathbf{\Omega}} dF_{\widetilde{\mathbf{X}}_K}(\mathbf{x}_K)$, which is the probability that a generated r.v. falls in $\mathbf{\Omega}$. Based on the results on Bernoulli trials, and following a similar discussion in [30], the variance of N_0 is $N_s\pi(1-\pi)$. When N_s is large, using s-normal approximation, the $1-\alpha$ confidence interval for π is

$$\frac{N_0}{N_s} - Z_{\frac{\alpha}{2}}\sqrt{\frac{\pi(1-\pi)}{N_s}} \le \pi \le \frac{N_0}{N_s} + Z_{\frac{\alpha}{2}}\sqrt{\frac{\pi(1-\pi)}{N_s}}. \tag{14}$$

It can be seen that the maximum width of the above confidence interval is achieved at $\pi = 0.5$. For example, when $N_s = 10\,000$, $\pi = 0.50$, and $\alpha = 0.05$, the confidence interval is $(N_0/N_s) \pm 0.01$. And the confidence interval will be much narrower for larger or smaller values of π.

In addition, note that the whole procedure we have proposed to evaluate the complex multidimensional integral in (11) of Step 2 can be considered as an importance sampling approach [30] to reduce the variance of the Monte Carlo estimation of integrals. We take advantage of the similarity between the shape of the nonnegative integrand function in (11), and the shape of a multivariate s-normal pdf by sampling from the multivariate s-normal distribution. It is known that the efficiency of the Monte Carlo method for integral evaluation can be improved by sampling from a distribution with a pdf of a similar shape to the absolute value of the integrand [30], [31].

4) Self-Improvement of Product Quality, and the Upper Bound of System Reliability: The dimension of the integral in (13) is $n \cdot (K+1)$, which is generally very large. Especially, the integral dimension depends on the production mission time t_K. It generally takes considerable computation resources to generate r.v. with such a large dimension. However, taking advantages of the properties of Gaussian r.v., computation resources for the situation that the product quality does not have self-improvement can be appreciably saved; this is discussed as follows.

If $q_j(t)$ is decreasing at t, then the product quality is self-improved at that time. Usually, the product quality of a manufacturing process does not have self-improvement unless the process is set up inappropriately so that degradation of MS components may even improve the system performance at certain times. When the product of an MMP does not have self-improvement, the evaluation of the system reliability can be much easier by taking advantages of the properties of Gaussian r.v. Lemma 2 is used to examine under what condition the product quality of an MMP does not have self-improvement.

Lemma 2: If all elements of $\mathbf{B}_j$, $j = 1, 2, \ldots, m$, in (4) are nonnegative, then the product quality of the MMP does not have self-improvement. Particularly, if $\mathbf{B}_j$, $j = 1, 2, \ldots, m$,

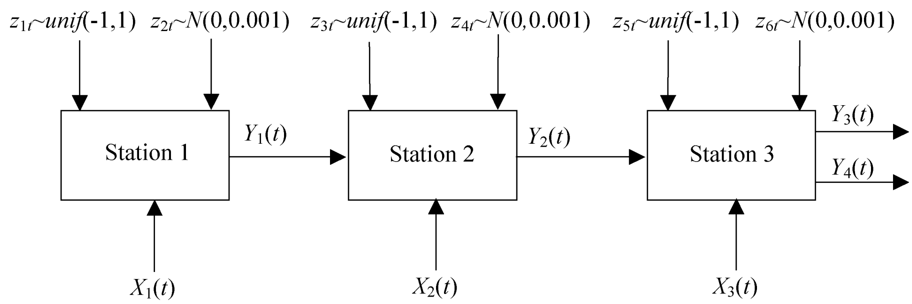

Fig. 5. A simple example of a three station manufacturing process.

are diagonal, the product quality of the MMP does not have self-improvement.

If $\mathbf{B}_j$ is a diagonal matrix, then it has only nonnegative elements, because $\mathbf{B}_j$ is positive semi-definite. The proof of Lemma 2 is obvious from (4), and Assumptions 4 & 5.

If the product quality of an MMP does not have self-improvement, then the event that $q_j(t)$ is within specification at any time by t_K is equivalent to the event that it is within specification at time t_K.

$$I_{t_K} \equiv \begin{cases} 1, & \mathbf{x}(t_K) \in \mathbf{\Omega}_K \\ 0, & \text{otherwise} \end{cases}$$

where $\mathbf{x}(t_K) \in \mathbf{\Omega}_K \Leftrightarrow \bigcap_{j=1}^{m} \{\mathbf{x}^T(t_K) \times \mathbf{B}_j \times \mathbf{x}(t_K) \leq a_j - d_j\}$.

If the product quality of an MMP does not have self-improvement, then

$$I_K = I_{t_K}.$$

Thus, (13) can be rewritten as

$$R(t_K) = \exp(-c \cdot t_K) \frac{\exp(-s_K)}{|\mathbf{\Sigma}_K|^{\frac{1}{2}}} \cdot |\widetilde{\mathbf{\Sigma}}_K|^{\frac{1}{2}} \cdot \int_{\mathbf{x}(t_K) \in \mathbf{\Omega}_K} dF_{\widetilde{\mathbf{X}}_K}(\mathbf{x}_K). \tag{15}$$

The integral domain of (15) depends only on $\mathbf{x}(t_K)$, not on $\mathbf{x}(t_{K-1}), \mathbf{x}(t_{K-2}), \ldots, \mathbf{x}(t_0)$. So the integral can be calculated based on the marginal distribution of $\widetilde{\mathbf{X}}(t_K)$, which is the last n elements of $\widetilde{\mathbf{X}}_K$. Because $\widetilde{\mathbf{X}}_K$ is multivariate Gaussian, the marginal distribution for $\widetilde{\mathbf{X}}(t_K)$ can be directly obtained from the distribution of $\widetilde{\mathbf{X}}_K$. Following this, the $n \cdot (K+1)$ dimension integral can be reduced to an n-dimension integral as in Result 4.

Result 4:

$$R(t_K) = \exp(-c \cdot t_K) \cdot \frac{\exp(-s_K)}{|\mathbf{\Sigma}_K|^{\frac{1}{2}}} \cdot |\widetilde{\mathbf{\Sigma}}_K|^{\frac{1}{2}} \cdot \int_{\mathbf{x}(t_K) \in \mathbf{\Omega}_K} dF_{\widetilde{\mathbf{X}}(t_K)}(\mathbf{x}(t_K)) \tag{16}$$

The proof of Result 4 is in Appendix A5. Obtaining the distribution of $\widetilde{\mathbf{X}}(t_K)$ in (16) is discussed in Appendix A6.

In Result 4, the dimension of $\widetilde{\mathbf{X}}(t_K)$ is n, which is independent of the mission time index K. If the distribution of $\widetilde{\mathbf{X}}(t_K)$ is known, evaluation of (16) requires the numerical calculation of an integral of n dimensions rather than $n(K+1)$ dimensions. Thus, (16) gives a simplified analytic solution of the system reliability for the case when the product quality has no self-improvement. This result is meaningful because the product quality of the majority of MMP does not have self-improvement. The multidimensional integral in (16) can be evaluated using a Monte Carlo method similar to that discussed in Section III-B-3.

In general cases when the product quality can have self-improvement, it is easy to see that $I_K = 1 \Rightarrow \mathbf{X}(t_K) \in \mathbf{\Omega}_K$, but the converse $(\mathbf{X}(t_K) \in \mathbf{\Omega} \Rightarrow I_K = 1)$ is not true generally. Equation (16) provides an upper bound estimation of the system reliability.

IV. Numerical Analysis, and Case Study

In this section, a simple numerical example is first provided to illustrate the procedures of QR-chain modeling & reliability evaluation. Then, a case study for a real automotive assembly process is introduced to demonstrate the effectiveness of the proposed approach.

A. Numerical Analysis

A manufacturing process with 3 stations, as in Fig. 5, is considered in the numerical analysis. There are 3 TE, which are the MS components. Let TE1, TE2, and TE3 denote the tooling elements at station 1, station 2, and station 3, respectively. The performance of these three TE is described by the process-variables $\mathbf{X}(t) \equiv [X_1(t)\ X_2(t)\ X_3(t)]^T$. The noise-variables z_{it}, $i = 1$, 2, 3, 4, 5, 6 (Fig. 5), interact with the process-variables, and impact the product quality. There are 4 product quality characteristics $Y_i(t)$, $i = 1, 2, 3, 4$. $Y_1(t)$, and $Y_2(t)$ are measurements on the outgoing products of station 1, and station 2, respectively. $Y_3(t)$ & $Y_4(t)$ are measurements on the final product.

1) The Process Model: The process model, (17)–(20), describes the effects of the noise variables & the process variables on the product quality characteristics

$$Y_1(t) = 0.274 \cdot X_1(t) + z_{2t} - 0.483 \cdot X_1(t) \cdot z_{1t} \tag{17}$$

$$\begin{aligned} Y_2(t) &= 0.372 \cdot Y_1(t) + 0.365 \cdot X_2(t) + 0.564 \cdot z_{4t} + X_2(t) \cdot z_{3t} \\ &= 0.102 \cdot X_1(t) + 0.365 \cdot X_2(t) + 0.372 \cdot z_{2t} \\ &\quad + 0.564 \cdot z_{4t} - 0.180 \cdot X_1(t) \cdot z_{1t} + X_2(t) \cdot z_{3t} \end{aligned} \tag{18}$$

$$\begin{aligned} Y_3(t) &= 0.628 \cdot Y_2(t) + 0.303 \cdot X_3(t) - 0.722 \cdot z_{6t} \\ &\quad + 0.775 \cdot X_3(t) \cdot z_{5t} \\ &= 0.064 \cdot X_1(t) + 0.229 \cdot X_2(t) + 0.303 \cdot X_3(t) \\ &\quad + 0.234 \cdot z_{2t} + 0.354 \cdot z_{4t} - 0.722 \cdot z_{6t} \\ &\quad - 0.113 \cdot X_1(t) \cdot z_{1t} + 0.628 \cdot X_2(t) \cdot z_{3t} \\ &\quad + 0.775 \cdot X_3(t) \cdot z_{5t} \end{aligned} \tag{19}$$

$$Y_4(t) = 0.426 \cdot X_3(t) - 0.703 \cdot X_3(t) \cdot z_{5t} \tag{20}$$

The distributions of z_{it}, $i = 1, 2, 3, 4, 5, 6$ are shown in Fig. 5. Refer to the general process model (1), for each $Y_j(t)$ in this example, $\eta_j = 0$. The $\boldsymbol{\alpha}_j$, $\boldsymbol{\beta}_j$, and $\boldsymbol{\Gamma}_j$ can be obtained by comparing (17)–(20) with (1) as

$$\boldsymbol{\Gamma}_1 = \begin{bmatrix} -0.483 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$
$$\boldsymbol{\Gamma}_2 = \begin{bmatrix} -0.180 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$
$$\boldsymbol{\Gamma}_3 = \begin{bmatrix} -0.113 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.628 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.775 & 0 \end{bmatrix}$$
$$\boldsymbol{\Gamma}_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.703 & 0 \end{bmatrix}$$
$$\begin{aligned} \boldsymbol{\alpha}_1 &= [0.274 \quad 0 \quad 0]^T \\ \boldsymbol{\alpha}_2 &= [0.102 \quad 0.365 \quad 0]^T \\ \boldsymbol{\alpha}_3 &= [0.064 \quad 0.229 \quad 0.303]^T \\ \boldsymbol{\alpha}_4 &= [0 \quad 0 \quad 0.426]^T \\ \boldsymbol{\beta}_1 &= [0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0]^T \\ \boldsymbol{\beta}_2 &= [0 \quad 0.372 \quad 0 \quad 0.564 \quad 0 \quad 0]^T \\ \boldsymbol{\beta}_3 &= [0 \quad 0.234 \quad 0 \quad 0.354 \quad 0 \quad -0.722]^T \\ \boldsymbol{\beta}_4 &= [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]^T. \end{aligned}$$

2) Manufacturing System Component Degradation Model: The following component degradation model is used for this MMP

$$\begin{aligned} \mathbf{X}(t_{k+1}) &= \mathbf{X}(t_k) + \boldsymbol{\varepsilon}_k \\ k &= 0, 1, 2, \ldots \boldsymbol{\varepsilon}_k \sim N(\boldsymbol{\mu}_\varepsilon, \mathbf{Q}) \\ \boldsymbol{\mu}_{\boldsymbol{\varepsilon}} &= 10^{-3} \cdot [2 \quad 2 \quad 2]^T \\ \mathbf{Q} &= 10^{-7} \cdot \begin{bmatrix} 2.5 & 0 & 0 \\ 0 & 2.5 & 0 \\ 0 & 0 & 2.5 \end{bmatrix} \\ h &= 1000. \end{aligned}$$

The initial value of the process-variables $\mathbf{X}(0) \sim N(\boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0)$, where $\boldsymbol{\mu}_0 = [0.1\ 0.1\ 0.1]^T$, and

$$\boldsymbol{\Sigma}_0 = \begin{bmatrix} 2.5 & 0 & 0 \\ 0 & 2.5 & 0 \\ 0 & 0 & 2.5 \end{bmatrix} \cdot 10^{-4}.$$

Compared with (3), $\mathbf{A}_k = \mathbf{I}$, $\mathbf{G}_k = \mathbf{I}$, $\forall k$.

3) Product Quality Assessment: Based on (4),

$$q_j\left(t|\mathbf{X}(t_k)\right) = \mathbf{X}(t_k)^T \times \mathbf{B}_j \times \mathbf{X}(t_k) + d_j, \quad j = 1, 2, 3, 4, \ t_k \le t < t_{k+1}$$

$\mathbf{B}_j$ & d_j can be obtained from (4), and the process model (17)–(20) as

$$\mathbf{B}_1 = \begin{bmatrix} 0.308 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
$$\mathbf{B}_2 = \begin{bmatrix} 0.043 & 0.037 & 0 \\ 0.037 & 1.133 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
$$\mathbf{B}_3 = \begin{bmatrix} 0.017 & 0.015 & 0.019 \\ 0.015 & 0.447 & 0.069 \\ 0.019 & 0.069 & 0.692 \end{bmatrix}$$
$$\mathbf{B}_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0.676 \end{bmatrix}$$
$$\begin{aligned} d_1 &= 10^{-3} \\ d_2 &= 0.457 \times 10^{-3} \\ d_3 &= 0.701 \times 10^{-3} \\ d_4 &= 0. \end{aligned}$$

The thresholds for $q_1(t|\mathbf{X}(t_k))$ & $q_2(t|\mathbf{X}(t_k))$ are ∞. The thresholds for $q_3(t|\mathbf{X}(t_k))$ & $q_4(t|\mathbf{X}(t_k))$ are both 0.04. That is, the quality specifications are only assigned for the product quality characteristics on the final product.

4) Component Catastrophic Failure: Due to the QR-chain effect, the failure rates of TE2, and TE3 are affected by $q_1(t)$, and $q_2(t)$, respectively. In this system

$$\begin{aligned} \lambda_1(t) &= 6 \times 10^{-7} \\ \lambda_2(t) &= 6 \times 10^{-7} + 3 \times 10^{-4} \cdot q_1(t) \\ \lambda_3(t) &= 6 \times 10^{-7} + 3 \times 10^{-4} \cdot q_2(t). \end{aligned}$$

Comparing this equation with (8)

$$\lambda_{01} = \lambda_{02} = \lambda_{03} = 6 \times 10^{-7} \quad \mathbf{s}_1 = [0 \quad 0 \quad 0 \quad 0]^T$$
$$\mathbf{s}_2 = [s \quad 0 \quad 0 \quad 0]^T \qquad \mathbf{s}_3 = [0 \quad s \quad 0 \quad 0]^T$$

$s = 3 \times 10^{-4}$ for this MMP.

5) System Reliability Evaluation: Because all elements of $\mathbf{B}_j$, $j = 1, 2, 3, 4$, are nonnegative, from Lemma 2 the product quality of the manufacturing process in Fig. 5 does not have self-improvement. Thus, Result 4 can be used to evaluate the system reliability. Fig. 6 shows the reliability plots calculated with Result 4. The following 3 cases are compared in Fig. 6(a).

Case i: $\boldsymbol{\varepsilon}_k = 0$, $s = 0$, i.e., the component degradation & the impact of product quality on MS component catastrophic failure rate are ignored.

Case ii: $\boldsymbol{\varepsilon}_k \neq 0$, $s = 0$, i.e., the component degradation is considered; but the impact of product quality on MS component catastrophic failure rate is ignored.

Case iii: $\boldsymbol{\varepsilon}_k \neq 0$, $s \neq 0$, i.e., both component degradation & the impact of product quality on MS component catastrophic failure rate are considered. Therefore, Case iii considers the QR-chain effect studied in this paper.

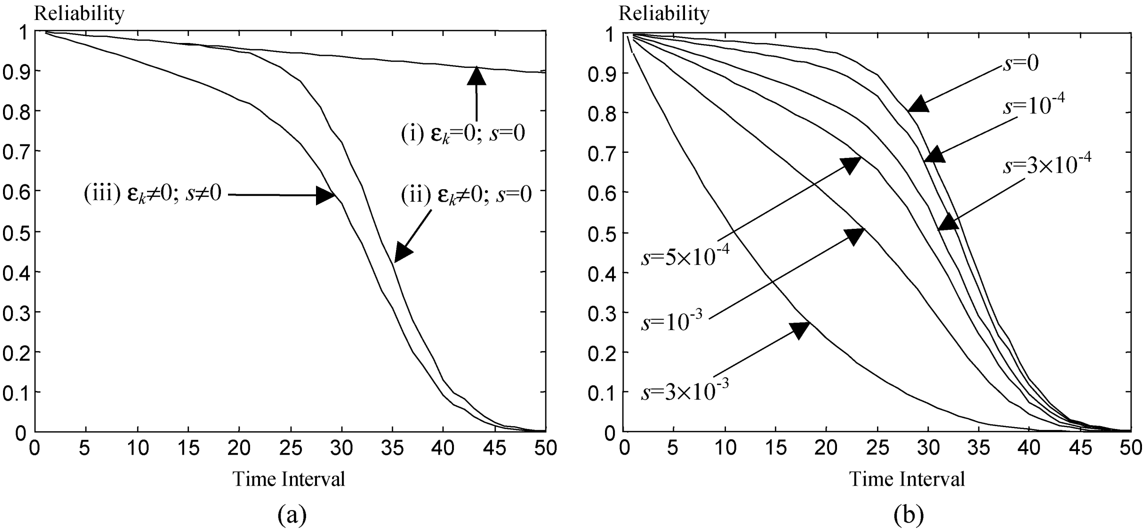

Fig. 6. System reliability & sensitivity analysis. (a) System reliability in three different cases; (b) sensitivity analysis for different s.

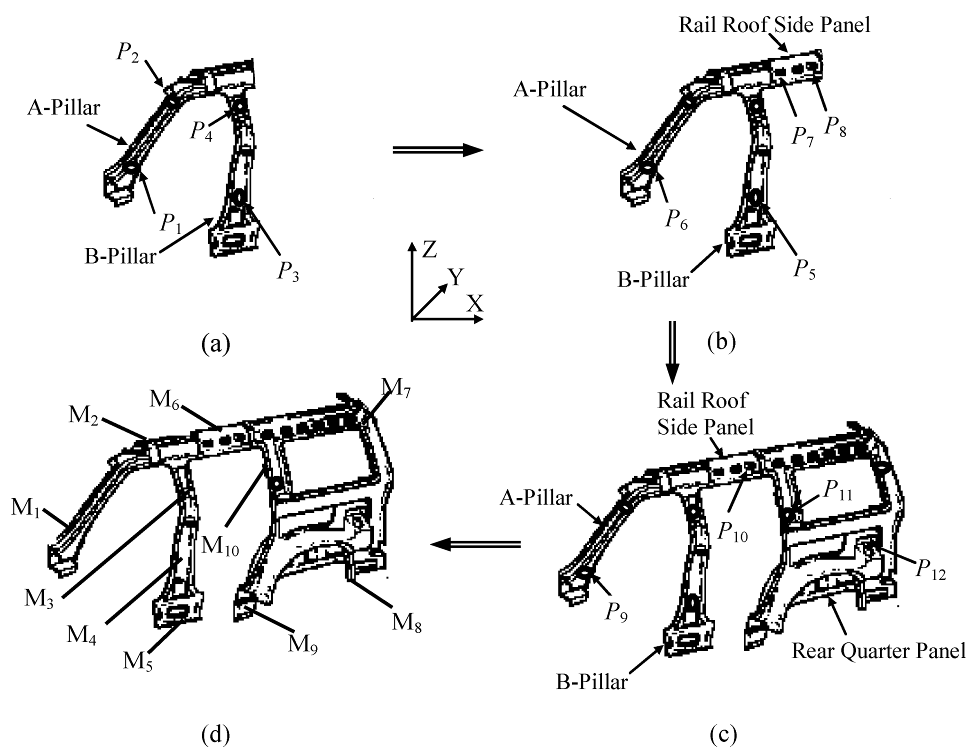

Fig. 7. A four station side-frame assembly process. (a) Station I; (b) Station II; (c) Station III; (d) Station IV: key quality characteristics.

Compared with Case iii, both Case i & Case ii tend to overestimate the system reliability because they incorrectly neglect the QR-chain effect existing in an MMP. If a scheduled maintenance policy is designed based on the results from Case i or Case ii, many unanticipated failures of the manufacturing process will occur due to the nonconforming product quality, or the impact from the incoming product quality of each station. A sensitivity analysis for various values of the QR-coefficient s is in Fig. 6(b).

B. Case Study

In an automotive body assembly process, the quality of the assembled part is determined by the positions & sizes of locating-holes on the sheet metal part, and the locating pins on the fixture. Fig. 7 is a four-station side frame assembly process. At Station I, two sheet metal parts, A-pillar & B-pillar, are assembled together. The subassembly of A-pillar & B-pillar is assembled with the rail roof side panel at Station II. Another part, the rear quarter panel, is added at Station III. The assembled side frame is inspected at Station IV. The MS components in this example are locating pins used to position the sheet metal parts at each station. Twelve locating pins are used in this process ($P_1 - P_{12}$ in Fig. 7). Ten PQC, which are dimensional deviations measured at Station IV (M1-M10 in Fig. 7), are used to define the product quality. The locating error due to the fixtures' tolerance or degradation at previous stations is reflected as the incoming locating-hole deviation/variation of the part at the current assembly station. The hole-center deviation

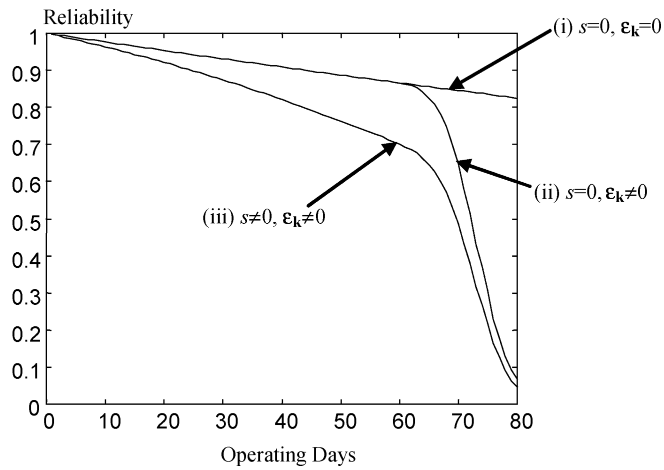

Fig. 8. System reliability of the side-frame assembly process.

of the incoming part impacts the locating pin broken rate of the current station. The combined effects of the locating-hole deviation of the sheet metal part & the locating pin degradation lead to outgoing part deviation of the current assembly station. Again, the assembled subassembly is used as the input of the next assembly station. As a result, there is a QR-co-effect at each station, and this QR-co-effect propagates from one station to the next station in an assembly process.

As an application of this paper, a QR-chain model has been built for this specific process in another paper [32] following exactly the procedure developed in this paper. A process model with the same form as (1) has been derived in [32] based on engineering analysis, such as kinematics analysis for sheet metal assembly processes. Engineering knowledge on locating pin & part hole interaction, locating pin wear, geometric layout of locators, and product design information are used in the engineering analysis in [32]. Due to limitation of space, we refer the readers to [32] for the details of this QR-chain model.

Based on the structure of the QR-chain model, and Lemma 2 of this paper, it has been found in [32] that the product quality of this process does not have self-improvement. Therefore, Result 4 can be used to evaluate the system reliability. Fig. 8 shows the reliability plots calculated based on Result 4. The Matlab code for the numerical evaluation of the system reliability was run on an IBM PC Pentium III machine. It takes about 10 seconds to evaluate a system reliability based on the approach developed in this paper. The three cases compared in Fig. 8 are the same as those compared in Fig. 6. Similar to the numerical example, the overestimation of system reliability when the QR-chain effect is overlooked can be seen in Fig. 8. In addition, it has been pointed out in [32] that this effect is consistent with the real failure data collected from automotive assembly processes.

APPENDIX

A. Derivation of Equation (4)

Based on the degradation model assumed in Section II-B, (1) can be rewritten as

$$Y_j(t) = \eta_j + \boldsymbol{\alpha}_j^T \times \mathbf{X}(t_k) + \boldsymbol{\beta}_j^T \times \mathbf{z}_t + \mathbf{X}(t_k)^T \times \boldsymbol{\Gamma}_j \times \mathbf{z}_t, \quad t_k \le t < t_{k+1}, \quad j = 1, 2, \ldots, m. \tag{21}$$

Under a given component degradation state $\mathbf{X}(t_k)$, $Y_j(t)$ is still a r.v. due to randomness of the noise-variable $\mathbf{z}_t$. The product quality, as a system performance index, is defined by the mean & variance of $Y_j(t)$ under a given component state $\mathbf{X}(t_k)$. Due to the term $\boldsymbol{\alpha}_j^T \times \mathbf{X}(t_k)$ in (21), the change of $\mathbf{X}(t_k)$ over time leads to a mean shift of $Y_j(t)$. Due to the term $\mathbf{X}(t_k)^T \times \boldsymbol{\Gamma}_j \times \mathbf{z}_t$ in (21), the change of $\mathbf{X}(t_k)$ can also lead to the change of the variability of $Y_j(t)$. Because variability is a very important quality index, it is important to include the interaction between the process-variable & the noise-variable in our model.

Quality is generally defined as the closeness of a quality characteristic to the target. To capture both the mean shift & the variability information, the product quality can be assessed as in Assumption 6. Following this concept, under given component degradation state $\mathbf{X}(t_k)$, $q_j(t)$ can be defined as

$$\begin{aligned} q_j\left(t|\mathbf{X}(t_k)\right) &\equiv E\left(\left(Y_j(t)-\gamma_j\right)^2 |\mathbf{X}(t_k)\right) \\ &= \mathrm{Var}(Y_j(t)|\mathbf{X}(t_k)) + E^2((Y_j(t)-\gamma_j)\,|\mathbf{X}(t_k)), \\ &\quad t_k \le t < t_{k+1} \end{aligned} \tag{22}$$

$\gamma_j \equiv$ the target value for the PQC.

Based on Assumption 4, the mean of the PQC achieves the target value when $\mathbf{X}(t_k) = \mathbf{0}$. Thus from (21)

$$\begin{aligned} \gamma_j &= E\left(Y_j(t)|\mathbf{X}(t_k)=\mathbf{0}\right) \\ &= \eta_j + \boldsymbol{\beta}_j^T \cdot E(\mathbf{z}_t); \end{aligned} \tag{23}$$

$$\begin{aligned} \mathrm{var}\left(Y_j(t)|\mathbf{X}(t_k)\right) &= \boldsymbol{\beta}_j^T \times \mathrm{cov}(\mathbf{z}_t) \times \boldsymbol{\beta}_j + \left(\mathbf{X}(t_k)^T \times \boldsymbol{\Gamma}_j\right) \\ &\quad \times \mathrm{cov}(\mathbf{z}_t) \times \left(\boldsymbol{\Gamma}_j^T \times \mathbf{X}(t_k)\right). \end{aligned} \tag{24}$$

From (21) & (23)

$$\begin{aligned} &E^2\left((Y_j(t)-\gamma_j)\,|\mathbf{X}(t_k)\right) \\ &\quad = \left(\boldsymbol{\alpha}_j^T \times \mathbf{X}(t_k) + \mathbf{X}(t_k)^T \times \boldsymbol{\Gamma}_j \times E(\mathbf{z}_t)\right)^2 \\ &\quad = \mathbf{X}(t_k)^T \times (\boldsymbol{\alpha}_j + \boldsymbol{\Gamma}_j \times E(\mathbf{z}_t)) \\ &\qquad \times (\boldsymbol{\alpha}_j + \boldsymbol{\Gamma}_j \times E(\mathbf{z}_t))^T \times \mathbf{X}(t_k) \end{aligned} \tag{25}$$

From (22), (24), & (25), the $q_j(t)$ is a quadratic function of $\mathbf{X}(t_k)$; thus

$$\begin{aligned} q_j\left(t|\mathbf{X}(t_k)\right) &= \mathbf{X}(t_k)^T \times \mathbf{B}_j \times \mathbf{X}(t_k) + d_j, \\ &\quad j = 1, 2, \ldots, m, t_k \le t < t_{k+1} \\ \mathbf{B}_j &\equiv \boldsymbol{\Gamma}_j \times \mathrm{cov}(\mathbf{z}_t) \times \boldsymbol{\Gamma}_j^T + (\boldsymbol{\alpha}_j + \boldsymbol{\Gamma}_j \times E(\mathbf{z}_t)) \\ &\quad \times (\boldsymbol{\alpha}_j + \boldsymbol{\Gamma}_j \times E(\mathbf{z}_t))^T \\ d_j &\equiv \boldsymbol{\beta}_j^T \times \mathrm{cov}(\mathbf{z}_t) \times \boldsymbol{\beta}_j. \end{aligned} \tag{26}$$

$\mathbf{B}_j$ in (26) is positive semi-definite, and $d_j \ge 0$.

B. Proof of Result 1

Conditioning on $\mathbf{X}_K$, the catastrophic failures of each component are s-independent following an NHPP. Furthermore, all components are connected in series. Thus,

$$\Pr\left\{E_{t_K}^c|\mathbf{X}_K\right\} = \exp\left(-\sum_{i=1}^{p}\left(\sum_{k=0}^{K-1}\lambda_i(t_k)\cdot h\right)\right) \tag{27}$$

From (4) & (8)

$$\lambda_i(t_k) = \lambda_{0i} + \mathbf{s}_i^T \times \mathbf{q}(t_k) = \lambda_{0i} + \mathbf{s}_i^T \times \mathbf{d} + \mathbf{X}(t_k)^T \times \left(\sum_{j=1}^{m} [\mathbf{s}_i]_j \cdot \mathbf{B}_j\right) \times \mathbf{X}(t_k), \quad i = 1,2,\ldots,p, \quad k = 0,1,\ldots,K.$$

So, $\sum_{i=1}^{p}(\sum_{k=0}^{K-1} \lambda_i(t_k) \cdot h) = \sum_{i=1}^{p}(\sum_{k=0}^{K-1}(\lambda_{0i} + \mathbf{s}_i^T \times \mathbf{d}) \cdot h) + \sum_{i=1}^{p}(\sum_{k=0}^{K-1} \mathbf{X}(k)^T \times (\sum_{j=1}^{m}[\mathbf{s}_i]_j \cdot \mathbf{B}_j) \times h \cdot \mathbf{X}(k))$Let $\mathbf{U}' \equiv h \cdot \sum_{i=1}^{p}\sum_{j=1}^{m}[\mathbf{s}_i]_j \cdot \mathbf{B}_j$. Because the elements of $\mathbf{s}_i$ are nonnegative, and $\mathbf{B}_j$, $j = 1,2,\ldots,m$ are positive semi-definite, it can be seen that $\mathbf{U}'$ is positive semi-definite. Also $\sum_{i=1}^{p}\sum_{k=0}^{K-1}(\lambda_{0i} + \mathbf{s}_i^T \times \mathbf{d}) \cdot h = c \cdot K \cdot h = c \cdot t_K$. Thus

$$\sum_{i=1}^{p}\left(\sum_{k=0}^{K-1} \lambda_i(t_k) \cdot h\right) = c \cdot t_K + \sum_{k=0}^{K-1} \mathbf{X}(t_k)^T \times \mathbf{U}' \times \mathbf{X}(t_k), \tag{28}$$

Finally, let

$$\mathbf{U}_K \equiv \begin{bmatrix} \mathbf{I}_{(K\times K)} \otimes \mathbf{U}' & \mathbf{0}_{(n\cdot K \times n)} \\ \mathbf{0}_{(n \times n\cdot K)} & \mathbf{0}_{(n\times n)} \end{bmatrix}$$

where $\mathbf{I}_{(K\times K)}$ is the $K \times K$ identity matrix. $\mathbf{U}_K$ is positive semi-definite. The $n \times n$ zero matrix in $\mathbf{U}_K$ is for $\mathbf{X}(t_K)$, which does not contribute to the catastrophic failure rates of components.

From (27) & (28)

$$\Pr\left\{E_{t_K}^c | \mathbf{X}_K\right\} = \exp\left(-\left(c \cdot t_K + \mathbf{X}_K^T \times \mathbf{U}_K \times \mathbf{X}_K\right)\right).$$

C. Derivation of Equation (11)

Un-conditioning on $\mathbf{X}_K$ by applying s-expectation to (10), then

$$R(t_K) = \underset{\mathbf{X}_K}{\mathrm{E}}\left[R(t_K|\mathbf{X}_K = \mathbf{x}_K)\right] = \int R(t_K|\mathbf{X}_K = \mathbf{x}_K) dF_{\mathbf{X}_K}(\mathbf{x}_K).$$

The $\int$ in this equation denotes a multidimensional integral. Further, from (10)

$$R(t_K) = \int \exp\left(-\left(c \cdot t_K + \mathbf{x}_K^T \times \mathbf{U}_K \times \mathbf{x}_K\right)\right) \cdot I_K dF_{\mathbf{X}_K}(\mathbf{x}_K) = \int_{\mathbf{x}_K \in \mathbf{\Omega}} \exp\left(-\left(c \cdot t_K + \mathbf{x}_K^T \times \mathbf{U}_K \times \mathbf{x}_K\right)\right) \times dF_{\mathbf{X}_K}(\mathbf{x}_K) \tag{29}$$

From (3), $\mathbf{X}(t_k)$ is Gaussian. Let $\mathbf{X}(t_k) \sim N(\boldsymbol{\mu}(k), \mathbf{\Sigma}(k))$, then $\boldsymbol{\mu}(k)$ & $\mathbf{\Sigma}(k)$ can be obtained recursively by

$$\boldsymbol{\mu}(k) = \mathbf{A}_{k-1} \times \boldsymbol{\mu}(k-1) + \mathbf{G}_{k-1} \times \boldsymbol{\mu}_\varepsilon, \quad k = 1,2,\ldots,K, \quad \boldsymbol{\mu}(0) = \boldsymbol{\mu}_0,$$
$$\mathbf{\Sigma}(k) = \mathbf{A}_{k-1} \times \mathbf{\Sigma}(k-1) \times \mathbf{A}_{k-1}^T + \mathbf{G}_{k-1} \times \mathbf{Q} \times \mathbf{G}_{k-1}^T, \quad k = 1,2,\ldots,K, \quad \mathbf{\Sigma}(0) = \mathbf{\Sigma}_0.$$

There exist constant matrices $\mathbf{H}(i)$, $i \geq 1$, so that

$$\mathbf{X}(t_{k+i}) = (\mathbf{A}_{k+i-1} \times \ldots \times \mathbf{A}_k) \times \mathbf{X}(t_k) + \mathbf{H}(i) \times \boldsymbol{\varepsilon}^{k,i}, \quad k = 1,2,\ldots,K, \quad i = 1,2,\ldots \tag{30}$$

where $\boldsymbol{\varepsilon}^{k,i} = [\boldsymbol{\varepsilon}_k^T \ \boldsymbol{\varepsilon}_{k+1}^T \ \ldots \ \boldsymbol{\varepsilon}_{k+i-1}^T]^T$. From (30)

$$\mathbf{\Sigma}(k+i,k) \equiv \mathrm{cov}\left(\mathbf{X}(t_{k+i}), \mathbf{X}(t_k)\right) = (\mathbf{A}_{k+i-1} \times \ldots \times \mathbf{A}_k) \times \mathbf{\Sigma}(k), \text{ and}$$
$$\mathbf{\Sigma}(k,k+i) = \mathbf{\Sigma}^T(k+i,k).$$

Let $\boldsymbol{\mu}_K \equiv [\boldsymbol{\mu}^T(0)\ \boldsymbol{\mu}^T(1)\ \ldots\ \boldsymbol{\mu}^T(K)]^T$

$$\mathbf{\Sigma}_K \equiv \begin{bmatrix} \mathbf{\Sigma}(0) & \mathbf{\Sigma}(0,1) & \ldots & \mathbf{\Sigma}(0,K) \\ \mathbf{\Sigma}(1,0) & \mathbf{\Sigma}(1) & \ldots & \mathbf{\Sigma}(1,K) \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{\Sigma}(K,0) & \mathbf{\Sigma}(K,1) & \ldots & \mathbf{\Sigma}(K) \end{bmatrix}.$$

Because $\mathbf{X}(t_0), \mathbf{X}(t_1), \ldots, \mathbf{X}(t_K)$ are jointly Gaussian, then

$$\mathbf{X}_K \sim N(\boldsymbol{\mu}_K, \mathbf{\Sigma}_K). \tag{31}$$

Based on (31), $F_{\mathbf{X}_K}(\mathbf{x}_K)$ is obtained, and substituted into (29), which yields

$$R(t_K) = \int_{\mathbf{x}_K \in \mathbf{\Omega}} \exp\left(-\left(c \cdot t_K + \mathbf{x}_K^T \times \mathbf{U}_K \times \mathbf{x}_K\right)\right) \times \frac{1}{(2\pi)^{\frac{n\cdot(K+1)}{2}} \cdot |\mathbf{\Sigma}_K|^{\frac{1}{2}}} \times \exp\left(-\frac{1}{2}(\mathbf{x}_K - \boldsymbol{\mu}_K)^T \times \mathbf{\Sigma}_K^{-1} \times (\mathbf{x}_K - \boldsymbol{\mu}_K)\right) d\mathbf{x}_K$$
$$= \exp(-c \cdot t_K) \int_{\mathbf{x}_K \in \mathbf{\Omega}} \frac{1}{(2\pi)^{\frac{n\cdot(K+1)}{2}} \cdot |\mathbf{\Sigma}_K|^{\frac{1}{2}}} \times \exp\left\{-\left(\mathbf{x}_K^T \times \mathbf{U}_K \times \mathbf{x}_K + \frac{1}{2}(\mathbf{x}_K - \boldsymbol{\mu}_K)^T \times \mathbf{\Sigma}_K^{-1} \times (\mathbf{x}_K - \boldsymbol{\mu}_K)\right)\right\} d\mathbf{x}_K \tag{32}$$

D. Proof of Lemma 1

$\mathbf{\Sigma}_K^{-1}$ is positive definite, and $\mathbf{U}_K$ is positive semi-definite (from Result 1) $\Rightarrow \widetilde{\mathbf{\Sigma}}_K^{-1} \equiv 2 \cdot \mathbf{U}_K + \mathbf{\Sigma}_K^{-1}$ is positive definite.

Let $\widetilde{\boldsymbol{\mu}}_K \equiv (\mathbf{U}_K + (\mathbf{\Sigma}_K^{-1}/2))^{-1} \times (\mathbf{\Sigma}_K^{-1}/2) \times \boldsymbol{\mu}_K$

$$\widetilde{\mathbf{\Sigma}}_K^{-1} \equiv 2 \cdot \mathbf{U}_K + \mathbf{\Sigma}_K^{-}$$
$$s_K \equiv \boldsymbol{\mu}_K^T \times \left(\frac{\mathbf{\Sigma}_K^{-1}}{2}\right) \times \boldsymbol{\mu}_K - \boldsymbol{\mu}_K^T \times \left[\left(\frac{\mathbf{\Sigma}_K^{-1}}{2}\right)^T \times \left(\left(\mathbf{U}_K + \left(\frac{\mathbf{\Sigma}_K^{-1}}{2}\right)\right)^{-1}\right)^T \times \left(\frac{\mathbf{\Sigma}_K^{-1}}{2}\right)\right] \times \boldsymbol{\mu}_K$$
$$= \boldsymbol{\mu}_K^T \times \left[\mathbf{U}_K^T \times \left(\mathbf{U}_K + \left(\frac{\mathbf{\Sigma}_K^{-1}}{2}\right)\right)^{-T} \times \left(\frac{\mathbf{\Sigma}_K^{-1}}{2}\right)\right] \times \boldsymbol{\mu}_K > 0.$$

Substitute $\widetilde{\boldsymbol{\mu}}_K$, $\widetilde{\boldsymbol{\Sigma}}_K^{-1}$, and s_K; and the equation

$$\frac{1}{2}(\mathbf{x}_K - \widetilde{\boldsymbol{\mu}}_K)^T \times \widetilde{\boldsymbol{\Sigma}}_K^{-1} \times (\mathbf{x}_K a - \widetilde{\boldsymbol{\mu}}_K) + s_K$$
$$= \mathbf{x}_K^T \times \mathbf{U}_K \times \mathbf{x}_K + \frac{1}{2}(\mathbf{x}_K - \boldsymbol{\mu}_K)^T \times \boldsymbol{\Sigma}_K^{-1} \times (\mathbf{x}_K - \boldsymbol{\mu}_K)$$

results.

E. Proof of Result 4

From (15)

$$R(t_K) = \exp(-c \cdot t_K) \cdot \frac{\exp(-s_K)}{|\boldsymbol{\Sigma}_K|^{\frac{1}{2}}} \cdot |\widetilde{\boldsymbol{\Sigma}}_K|^{\frac{1}{2}} \cdot \int_{\mathbf{x}(t_K) \in \boldsymbol{\Omega}_K} dF_{\widetilde{\mathbf{X}}_K}(\mathbf{x}_K)$$
$$= \exp(-c \cdot t_K) \cdot \frac{\exp(-s_K)}{|\boldsymbol{\Sigma}_K|^{\frac{1}{2}}} \cdot |\widetilde{\boldsymbol{\Sigma}}_K|^{\frac{1}{2}} \cdot \int \Pr\left\{\widetilde{\mathbf{X}}(t_K) \in \boldsymbol{\Omega}_K | \widetilde{\mathbf{X}}_K = \mathbf{x}_K\right\} dF_{\widetilde{\mathbf{X}}_K}(\mathbf{x}_K)$$
$$= \exp(-c \cdot t_K) \cdot \frac{\exp(-s_K)}{|\boldsymbol{\Sigma}_K|^{\frac{1}{2}}} \cdot |\widetilde{\boldsymbol{\Sigma}}_K|^{\frac{1}{2}} \cdot \mathop{\mathrm{E}}_{\widetilde{\mathbf{X}}_K}\left[\Pr\left\{\widetilde{\mathbf{X}}(t_K) \in \boldsymbol{\Omega}_K | \widetilde{\mathbf{X}}_K\right\}\right]$$
$$= \exp(-c \cdot t_K) \cdot \frac{\exp(-s_K)}{|\boldsymbol{\Sigma}_K|^{\frac{1}{2}}} \cdot |\widetilde{\boldsymbol{\Sigma}}_K|^{\frac{1}{2}} \cdot \Pr\left\{\widetilde{\mathbf{X}}(t_K) \in \boldsymbol{\Omega}_K\right\}$$
$$= \exp(-c \cdot t_K) \cdot \frac{\exp(-s_K)}{|\boldsymbol{\Sigma}_K|^{\frac{1}{2}}} \cdot |\widetilde{\boldsymbol{\Sigma}}_K|^{\frac{1}{2}} \cdot \int_{\mathbf{x}(t_K) \in \boldsymbol{\Omega}_K} dF_{\widetilde{\mathbf{X}}(t_K)}(\mathbf{x}(t_K))$$

F. Distribution of $\widetilde{\mathbf{X}}(t_K)$

Based on the property of multivariate Gaussian r.v., $\widetilde{\mathbf{X}}(t_K)$ is Gaussian with $\widetilde{\mathbf{X}}(t_K) \sim N(\widetilde{\boldsymbol{\mu}}(K), \widetilde{\boldsymbol{\Sigma}}(K))$, where $\widetilde{\boldsymbol{\mu}}(K) \equiv E(\widetilde{\mathbf{X}}(t_K))$, and $\widetilde{\boldsymbol{\Sigma}}(K) \equiv \mathrm{cov}(\widetilde{\mathbf{X}}(t_K))$. Partition $\widetilde{\boldsymbol{\Sigma}}_K$, $\widetilde{\boldsymbol{\mu}}_K$ as

$$\widetilde{\boldsymbol{\Sigma}}_K = \left[\begin{array}{c|c} \underset{(n \cdot K \times n \cdot K)}{\boldsymbol{\Sigma}11} & \underset{(n \cdot K \times n)}{\boldsymbol{\Sigma}12} \\ \hline \underset{(n \times n \cdot K)}{\boldsymbol{\Sigma}21} & \underset{(n \times n)}{\boldsymbol{\Sigma}22} \end{array}\right] \quad \widetilde{\boldsymbol{\mu}}_K = \left[\begin{array}{c} \underset{(n \cdot K \times 1)}{\boldsymbol{\mu}1} \\ \hline \underset{(n \times 1)}{\boldsymbol{\mu}2} \end{array}\right].$$

It is easy to see that $\widetilde{\boldsymbol{\mu}}(K) = \boldsymbol{\mu}2$, and $\widetilde{\boldsymbol{\Sigma}}(K) = \boldsymbol{\Sigma}22$. So,

$$dF_{\widetilde{\mathbf{X}}(t_K)}(\mathbf{x}(t_K)) = \frac{1}{(2\pi)^{\frac{n}{2}} \cdot \left|\widetilde{\boldsymbol{\Sigma}}(K)\right|^{\frac{1}{2}}} \times \exp\left(-\frac{1}{2}(\mathbf{x}(t_K) - \widetilde{\boldsymbol{\mu}}(K))^T \times \widetilde{\boldsymbol{\Sigma}}(K)^{-1}(\mathbf{x}(t_K) - \widetilde{\boldsymbol{\mu}}(K))\right) d\mathbf{x}(t_K)$$

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